

AN INVARIANT THEORY OF SPACELIKE SURFACES IN THE FOUR-DIMENSIONAL MINKOWSKI SPACE

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ABSTRACT. We consider spacelike surfaces in the four-dimensional Minkowski space and introduce geometrically an invariant linear map of Weingarten-type in the tangent plane at any point of the surface under consideration. This allows us to introduce principal lines and an invariant moving frame field. Writing derivative formulas of Frenet-type for this frame field, we obtain eight invariant functions. We prove a fundamental theorem of Bonnet-type, stating that these eight invariants under some natural conditions determine the surface up to a motion.

We show that the basic geometric classes of spacelike surfaces in the four-dimensional Minkowski space, determined by conditions on their invariants, can be interpreted in terms of the properties of the two geometric figures: the tangent indicatrix, and the normal curvature ellipse.

We apply our theory to a class of spacelike general rotational surfaces.

1. INTRODUCTION

In this paper we consider the general theory of spacelike surfaces in the four-dimensional Minkowski space \mathbb{R}_1^4 . The basic feature of our approach to this theory is the introduction of an invariant linear map of Weingarten-type in the tangent plane at any point of the surface. Studying surfaces in the Euclidean space \mathbb{R}^4 , in [2] we introduced a linear map of Weingarten-type, which plays a similar role in the theory of surfaces in \mathbb{R}^4 as the Weingarten map in the theory of surfaces in \mathbb{R}^3 . We gave a geometric interpretation of the second fundamental form and the Weingarten map of the surface in [5]. Following our approach to the surfaces in \mathbb{R}^4 , here we develop the theory of spacelike surfaces in \mathbb{R}_1^4 in a similar way.

Let M^2 be a spacelike surface in \mathbb{R}_1^4 . Considering the tangent space $T_p M^2$ at a point $p \in M^2$, we introduce an invariant ζ_{g_1, g_2} of a pair of two tangents g_1, g_2 using the second fundamental tensor σ of M^2 . By means of this invariant we define conjugate, asymptotic, and principal tangents.

The second fundamental form II of the surface M^2 at a point $p \in M^2$ is introduced on the base of conjugacy of two tangents at the point. The second fundamental form II determines an invariant linear map of Weingarten-type $\gamma : T_p M^2 \rightarrow T_p M^2$ at any point of M^2 in the standard way. The map γ generates two invariants k and \varkappa . We prove that the invariant \varkappa is the curvature of the normal connection of the surface. The number of asymptotic tangents at a point of M^2 is determined by the sign of the invariant k . In the case $k = 0$ there exists a one-parameter family of asymptotic lines, which are principal. It is interesting to note that the "umbilical" points, i.e. points at which the coefficients of the first and the second fundamental forms are proportional, are exactly the points at which the mean curvature vector H is zero. Minimal spacelike surfaces are characterized by the equality $\varkappa^2 - k = 0$.

Analogously to \mathbb{R}^3 , the invariants k and \varkappa divide the points of M^2 into four types: flat, elliptic, hyperbolic and parabolic points. The surfaces consisting of flat points are characterized by the conditions $k = \varkappa = 0$. In Section 4 we give a local geometric description of

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spacelike surfaces consisting of flat points whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. We prove that:

Any spacelike surface consisting of flat points whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector either lies in a hyperplane of \mathbb{R}_1^4 or is part of a developable ruled surface in \mathbb{R}_1^4 .

We introduce the indicatrix of Dupin χ at an arbitrary (non-flat) point of a spacelike surface in \mathbb{R}_1^4 by means of the second fundamental form as in the theory of surfaces in \mathbb{R}^3 . Then the elliptic, hyperbolic and parabolic points of a spacelike surface M^2 in \mathbb{R}_1^4 are characterized in terms of the indicatrix χ as in \mathbb{R}^3 . The conjugacy in terms of the second fundamental form coincides with the conjugacy with respect to the indicatrix χ . In Section 3 we prove that:

A spacelike surface M^2 is minimal if and only if the indicatrix χ is a circle.

A spacelike surface M^2 is with flat normal connection if and only if the indicatrix χ is a rectangular hyperbola (a Lorentz circle).

In the local theory of surfaces in Euclidean space a statement of significant importance is a theorem of Bonnet-type giving the natural conditions under which the surface is determined up to a motion. A theorem of this type was proved for surfaces with flat normal connection by B.-Y. Chen in [1]. In [3] we proved a fundamental theorem of Bonnet-type for minimal surfaces in \mathbb{R}^4 and in [4] we proved such a theorem for surfaces in \mathbb{R}^4 free of minimal points. Here we consider spacelike surfaces in \mathbb{R}_1^4 whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector. Using a geometrically determined moving frame of Frenet-type on such a surface and the corresponding derivative formulas, we obtain eight invariant functions. In Section 5 and Section 6 we prove our basic Theorem 5.1 and Theorem 6.1, stating that

Any spacelike surface whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector is determined up to a motion in \mathbb{R}_1^4 by its eight invariant functions satisfying some natural conditions.

In Section 7 we apply our theory to a class of spacelike general rotational surfaces in \mathbb{R}_1^4 .

2. INVARIANTS OF A TANGENT ON A SPACELIKE SURFACE IN \mathbb{R}_1^4

We consider the Minkowski space \mathbb{R}_1^4 endowed with the metric \langle, \rangle of signature $(3, 1)$. Let $Oe_1e_2e_3e_4$ be a fixed orthonormal coordinate system in \mathbb{R}_1^4 , i.e. $e_1^2 = e_2^2 = e_3^2 = 1$, $e_4^2 = -1$, giving the orientation of \mathbb{R}_1^4 .

A surface $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}$ ($\mathcal{D} \subset \mathbb{R}^2$) in \mathbb{R}_1^4 is said to be *spacelike* if \langle, \rangle induces a Riemannian metric g on M^2 . Thus at each point p of a spacelike surface M^2 we have the following decomposition:

$$\mathbb{R}_1^4 = T_p M^2 \oplus N_p M^2$$

with the property that the restriction of the metric \langle, \rangle onto the tangent space $T_p M^2$ is of signature $(2, 0)$, and the restriction of the metric \langle, \rangle onto the normal space $N_p M^2$ is of signature $(1, 1)$.

Let M^2 be a spacelike surface in \mathbb{R}_1^4 . The tangent space to M^2 at an arbitrary point $p = z(u, v)$ of M^2 is $\text{span}\{z_u, z_v\}$, where $\langle z_u, z_u \rangle > 0$, $\langle z_v, z_v \rangle > 0$. We use the standard denotations $E(u, v) = \langle z_u, z_u \rangle$, $F(u, v) = \langle z_u, z_v \rangle$, $G(u, v) = \langle z_v, z_v \rangle$ for the coefficients of the first fundamental form

$$I(\lambda, \mu) := E\lambda^2 + 2F\lambda\mu + G\mu^2, \quad \lambda, \mu \in \mathbb{R}.$$

Since $I(\lambda, \mu)$ is positive definite we set $W = \sqrt{EG - F^2}$.

We choose a normal frame field $\{n_1, n_2\}$ such that $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$, and the quadruple $\{z_u, z_v, n_1, n_2\}$ is positively oriented in \mathbb{R}_1^4 . Denote by ∇' the standard covariant

derivative in \mathbb{R}_1^4 and consider the functions

$$\begin{aligned} c_{11}^1 &= \langle z_{uu}, n_1 \rangle; & c_{11}^2 &= \langle z_{uu}, n_2 \rangle; \\ c_{12}^1 &= \langle z_{uv}, n_1 \rangle; & c_{12}^2 &= \langle z_{uv}, n_2 \rangle; \\ c_{22}^1 &= \langle z_{vv}, n_1 \rangle; & c_{22}^2 &= \langle z_{vv}, n_2 \rangle. \end{aligned}$$

Then we have the following derivative formulas:

$$\begin{aligned} \nabla'_{z_u} z_u &= z_{uu} = \Gamma_{11}^1 z_u + \Gamma_{11}^2 z_v + c_{11}^1 n_1 - c_{11}^2 n_2; \\ \nabla'_{z_u} z_v &= z_{uv} = \Gamma_{12}^1 z_u + \Gamma_{12}^2 z_v + c_{12}^1 n_1 - c_{12}^2 n_2; \\ \nabla'_{z_v} z_v &= z_{vv} = \Gamma_{22}^1 z_u + \Gamma_{22}^2 z_v + c_{22}^1 n_1 - c_{22}^2 n_2, \end{aligned}$$

where Γ_{ij}^k are the Christoffel's symbols. If σ denotes the second fundamental tensor of M^2 , then we have

$$\begin{aligned} \sigma(z_u, z_u) &= c_{11}^1 n_1 - c_{11}^2 n_2, \\ \sigma(z_u, z_v) &= c_{12}^1 n_1 - c_{12}^2 n_2, \\ \sigma(z_v, z_v) &= c_{22}^1 n_1 - c_{22}^2 n_2. \end{aligned} \tag{2.1}$$

Obviously, the surface M^2 lies in a 2-plane if and only if M^2 is totally geodesic, i.e. $c_{ij}^k = 0$, $i, j, k = 1, 2$. So, we assume that at least one of the coefficients c_{ij}^k is not zero.

We shall define conjugate tangents at any point of the surface M^2 .

Let g be a tangent at the point $p \in M^2$ determined by the non-zero vector $X = \lambda z_u + \mu z_v$. We consider the map $\sigma_g : T_p M^2 \rightarrow N_p M^2$, defined by

$$\sigma_g(Y) = \sigma \left(\frac{\lambda z_u + \mu z_v}{\sqrt{I(\lambda, \mu)}}, Y \right), \quad Y \in T_p M^2. \tag{2.2}$$

Obviously σ_g is a linear map, which does not depend on the choice of the non-zero vector X collinear with g . Using (2.1) and (2.2) we obtain the following decomposition of the normal vectors $\sigma_g(z_u)$ and $\sigma_g(z_v)$:

$$\begin{aligned} \sigma_g(z_u) &= \frac{\lambda c_{11}^1 + \mu c_{12}^1}{\sqrt{I(\lambda, \mu)}} n_1 - \frac{\lambda c_{11}^2 + \mu c_{12}^2}{\sqrt{I(\lambda, \mu)}} n_2, \\ \sigma_g(z_v) &= \frac{\lambda c_{12}^1 + \mu c_{22}^1}{\sqrt{I(\lambda, \mu)}} n_1 - \frac{\lambda c_{12}^2 + \mu c_{22}^2}{\sqrt{I(\lambda, \mu)}} n_2. \end{aligned} \tag{2.3}$$

Let $g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v$ and $g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v$ be two tangents at the point $p \in M^2$. We consider the parallelograms determined by the pairs of normal vectors $\sigma_{g_1}(z_u)$, $\sigma_{g_2}(z_v)$ and $\sigma_{g_2}(z_u)$, $\sigma_{g_1}(z_v)$ in the Lorentz plane $\text{span}\{n_1, n_2\}$. The oriented areas of these parallelograms are denoted by $S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))$, and $S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))$, respectively. We assign the quantity ζ_{g_1, g_2} to the pair of tangents g_1, g_2 , defined by

$$\zeta_{g_1, g_2} = \frac{S(\sigma_{g_1}(z_u), \sigma_{g_2}(z_v))}{W} + \frac{S(\sigma_{g_2}(z_u), \sigma_{g_1}(z_v))}{W}. \tag{2.4}$$

Using equalities (2.3) we calculate that

$$\zeta_{g_1, g_2} = \frac{2 \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix} \lambda_1 \lambda_2 + \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{21}^1 & c_{22}^1 \end{vmatrix} (\lambda_1 \mu_2 + \mu_1 \lambda_2) + 2 \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix} \mu_1 \mu_2}{W \sqrt{I(\lambda_1, \mu_1)} \sqrt{I(\lambda_2, \mu_2)}}.$$

We introduce the following functions:

$$\Delta_1 = \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad \Delta_2 = \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad \Delta_3 = \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix};$$

$$L(u, v) = \frac{2\Delta_1}{W}, \quad M(u, v) = \frac{\Delta_2}{W}, \quad N(u, v) = \frac{2\Delta_3}{W}.$$

Hence, ζ_{g_1, g_2} is expressed as follows:

$$(2.5) \quad \zeta_{g_1, g_2} = \frac{L\lambda_1\lambda_2 + M(\lambda_1\mu_2 + \mu_1\lambda_2) + N\mu_1\mu_2}{\sqrt{I(\lambda_1, \mu_1)}\sqrt{I(\lambda_2, \mu_2)}}.$$

Proposition 2.1. *The quantity ζ_{g_1, g_2} is invariant under any change of the parameters on M^2 .*

Proof: Let

$$(2.6) \quad \begin{aligned} u &= u(\bar{u}, \bar{v}); \\ v &= v(\bar{u}, \bar{v}), \end{aligned} \quad (\bar{u}, \bar{v}) \in \bar{\mathcal{D}}, \quad \bar{\mathcal{D}} \subset \mathbb{R}^2$$

be a smooth change of the parameters (u, v) on M^2 with $J = u_{\bar{u}} v_{\bar{v}} - u_{\bar{v}} v_{\bar{u}} \neq 0$. Then

$$\begin{aligned} z_{\bar{u}} &= z_u u_{\bar{u}} + z_v v_{\bar{u}}, \\ z_{\bar{v}} &= z_u u_{\bar{v}} + z_v v_{\bar{v}}. \end{aligned}$$

If $\bar{E} = \langle z_{\bar{u}}, z_{\bar{u}} \rangle$, $\bar{F} = \langle z_{\bar{u}}, z_{\bar{v}} \rangle$ and $\bar{G} = \langle z_{\bar{v}}, z_{\bar{v}} \rangle$, then we have

$$\begin{aligned} \bar{E} &= u_{\bar{u}}^2 E + 2u_{\bar{u}} v_{\bar{u}} F + v_{\bar{u}}^2 G, \\ \bar{F} &= u_{\bar{u}} u_{\bar{v}} E + (u_{\bar{u}} v_{\bar{v}} + v_{\bar{u}} u_{\bar{v}}) F + v_{\bar{u}} v_{\bar{v}} G, \\ \bar{G} &= u_{\bar{v}}^2 E + 2u_{\bar{v}} v_{\bar{v}} F + v_{\bar{v}}^2 G \end{aligned}$$

and $\bar{E}\bar{G} - \bar{F}^2 = J^2 (EG - F^2)$, hence $\bar{W} = \varepsilon J W$, $\varepsilon = \text{sign } J$.

Let

$$\begin{aligned} \sigma(z_{\bar{u}}, z_{\bar{u}}) &= \bar{c}_{11}^1 n_1 - \bar{c}_{11}^2 n_2, \\ \sigma(z_{\bar{u}}, z_{\bar{v}}) &= \bar{c}_{12}^1 n_1 - \bar{c}_{12}^2 n_2, \\ \sigma(z_{\bar{v}}, z_{\bar{v}}) &= \bar{c}_{22}^1 n_1 - \bar{c}_{22}^2 n_2. \end{aligned}$$

Then from (2.6) and (2.1) we find

$$\begin{aligned} \bar{c}_{11}^k &= u_{\bar{u}}^2 c_{11}^k + 2u_{\bar{u}} v_{\bar{u}} c_{12}^k + v_{\bar{u}}^2 c_{22}^k, \\ \bar{c}_{12}^k &= u_{\bar{u}} u_{\bar{v}} c_{11}^k + (u_{\bar{u}} v_{\bar{v}} + u_{\bar{v}} v_{\bar{u}}) c_{12}^k + v_{\bar{u}} v_{\bar{v}} c_{22}^k, \\ \bar{c}_{22}^k &= u_{\bar{v}}^2 c_{11}^k + 2u_{\bar{v}} v_{\bar{v}} c_{12}^k + v_{\bar{v}}^2 c_{22}^k. \end{aligned} \quad (k = 1, 2),$$

and hence

$$\begin{aligned} \bar{\Delta}_1 &= J (u_{\bar{u}}^2 \Delta_1 + u_{\bar{u}} v_{\bar{u}} \Delta_2 + v_{\bar{u}}^2 \Delta_3); \\ \bar{\Delta}_2 &= J (2u_{\bar{u}} u_{\bar{v}} \Delta_1 + (u_{\bar{u}} v_{\bar{v}} + u_{\bar{v}} v_{\bar{u}}) \Delta_2 + 2v_{\bar{u}} v_{\bar{v}} \Delta_3); \\ \bar{\Delta}_3 &= J (u_{\bar{v}}^2 \Delta_1 + u_{\bar{v}} v_{\bar{v}} \Delta_2 + v_{\bar{v}}^2 \Delta_3). \end{aligned}$$

Thus we find that the functions \bar{L} , \bar{M} , \bar{N} are expressed as follows:

$$(2.7) \quad \begin{aligned} \bar{L} &= \varepsilon(u_{\bar{u}}^2 L + 2 u_{\bar{u}} v_{\bar{u}} M + v_{\bar{u}}^2 N), \\ \bar{M} &= \varepsilon(u_{\bar{u}} u_{\bar{v}} L + (u_{\bar{u}} v_{\bar{v}} + v_{\bar{u}} u_{\bar{v}}) M + v_{\bar{u}} v_{\bar{v}} N), \\ \bar{N} &= \varepsilon(u_{\bar{v}}^2 L + 2 u_{\bar{v}} v_{\bar{v}} M + v_{\bar{v}}^2 N). \end{aligned}$$

Hence, the functions L, M, N change in the same way as the coefficients of the first fundamental form E, F, G under any change of the parameters on M^2 .

If $X = \lambda z_u + \mu z_v = \bar{\lambda} z_{\bar{u}} + \bar{\mu} z_{\bar{v}}$, then $\lambda = u_{\bar{u}} \bar{\lambda} + u_{\bar{v}} \bar{\mu}$, $\mu = v_{\bar{u}} \bar{\lambda} + v_{\bar{v}} \bar{\mu}$. Using (2.7) we obtain

$$\bar{L} \bar{\lambda}_1 \bar{\lambda}_2 + \bar{M} (\bar{\lambda}_1 \bar{\mu}_2 + \bar{\mu}_1 \bar{\lambda}_2) + \bar{N} \bar{\mu}_1 \bar{\mu}_2 = \varepsilon (L \lambda_1 \lambda_2 + M (\lambda_1 \mu_2 + \mu_1 \lambda_2) + N \mu_1 \mu_2).$$

Having in mind that $\bar{I}(\bar{\lambda}, \bar{\mu}) = I(\lambda, \mu)$, we get

$$\bar{\zeta}_{g_1, g_2} = \frac{\bar{L} \bar{\lambda}_1 \bar{\lambda}_2 + \bar{M} (\bar{\lambda}_1 \bar{\mu}_2 + \bar{\mu}_1 \bar{\lambda}_2) + \bar{N} \bar{\mu}_1 \bar{\mu}_2}{\sqrt{I(\bar{\lambda}_1, \bar{\mu}_1)} \sqrt{I(\bar{\lambda}_2, \bar{\mu}_2)}} = \varepsilon \frac{L \lambda_1 \lambda_2 + M (\lambda_1 \mu_2 + \mu_1 \lambda_2) + N \mu_1 \mu_2}{\sqrt{I(\lambda_1, \mu_1)} \sqrt{I(\lambda_2, \mu_2)}} = \varepsilon \zeta_{g_1, g_2}.$$

Consequently, ζ_{g_1, g_2} is invariant (up to the orientation of the tangent space or the normal space of the surface). \square

Definition 2.2. Two tangents $g_1 : X_1 = \lambda_1 z_u + \mu_1 z_v$ and $g_2 : X_2 = \lambda_2 z_u + \mu_2 z_v$ are said to be *conjugate tangents* if $\zeta_{g_1, g_2} = 0$.

Obviously, $\zeta_{g_1, g_2} = 0$ if and only if

$$L \lambda_1 \lambda_2 + M (\lambda_1 \mu_2 + \lambda_2 \mu_1) + N \mu_1 \mu_2 = 0.$$

The last formula gives us the idea to define second fundamental form II of the surface M^2 at $p \in M^2$ as follows. Let $X = \lambda z_u + \mu z_v$, $(\lambda, \mu) \neq (0, 0)$ be a tangent vector at a point $p \in M^2$. Then

$$II(\lambda, \mu) = L \lambda^2 + 2 M \lambda \mu + N \mu^2, \quad \lambda, \mu \in \mathbb{R}.$$

We have already seen in the proof of Proposition 2.1 that the functions L, M, N change in the same way as the coefficients E, F, G under any change of the parameters on M^2 . If $\{\tilde{n}_1, \tilde{n}_2\}$ is another normal frame field of M^2 , such that $\langle \tilde{n}_1, \tilde{n}_1 \rangle = 1$, $\langle \tilde{n}_1, \tilde{n}_2 \rangle = 0$, $\langle \tilde{n}_2, \tilde{n}_2 \rangle = -1$, then

$$\begin{aligned} n_1 &= \varepsilon' (\cosh \theta \tilde{n}_1 + \sinh \theta \tilde{n}_2); \\ n_2 &= \varepsilon' (\sinh \theta \tilde{n}_1 + \cosh \theta \tilde{n}_2); \end{aligned} \quad \varepsilon' = \pm 1.$$

The relation between the corresponding functions c_{ij}^k and \tilde{c}_{ij}^k , $i, j, k = 1, 2$ is given by the equalities

$$\begin{aligned} \tilde{c}_{ij}^1 &= \varepsilon' (\cosh \theta c_{ij}^1 - \sinh \theta c_{ij}^2); \\ \tilde{c}_{ij}^2 &= \varepsilon' (-\sinh \theta c_{ij}^1 + \cosh \theta c_{ij}^2); \end{aligned} \quad i, j = 1, 2.$$

Thus, $\tilde{\Delta}_i = \Delta_i$, $i = 1, 2, 3$, and $\tilde{L} = L$, $\tilde{M} = M$, $\tilde{N} = N$. So, the functions L, M, N do not depend on the normal frame of the surface. Hence, the second fundamental form II is invariant up to the orientation of the tangent space or the normal space of the surface.

As in the classical differential geometry of surfaces in \mathbb{R}^3 and in the same way as in the theory of surfaces on \mathbb{R}^4 [4], the second fundamental form II determines conjugate tangents at a point p of M^2 . The considerations above show that the conjugacy in terms of the second fundamental form is the conjugacy defined by the invariant ζ_{g_1, g_2} .

We shall assign two invariants ν_g and α_g to any tangent g of the surface in the following way. Let $g : X = \lambda z_u + \mu z_v$ be a tangent and g^\perp be its orthogonal tangent, determined by

the vector

$$(2.8) \quad X^\perp = -\frac{F\lambda + G\mu}{W} z_u + \frac{E\lambda + F\mu}{W} z_v.$$

We define

$$\nu_g = \zeta_{g,g}; \quad \alpha_g = \zeta_{g,g^\perp}.$$

We call ν_g the *normal curvature* of the tangent g , and α_g - the *geodesic torsion* of g .

Equalities (2.4) and (2.5) imply that

$$\nu_g = 2 \frac{S(\sigma_g(z_u), \sigma_g(z_v))}{W} = \frac{II(\lambda, \mu)}{I(\lambda, \mu)}.$$

Hence, the normal curvature of the tangent g is two times the oriented area of the parallelogram determined by the normal vectors $\sigma_g(z_u)$ and $\sigma_g(z_v)$. The invariant ν_g is expressed by the first and the second fundamental forms of the surface in the same way as the normal curvature of a tangent in the theory of surfaces in \mathbb{R}^3 .

Using (2.5) and (2.8) we get

$$\alpha_g = \frac{\lambda^2(EM - FL) + \lambda\mu(EN - GL) + \mu^2(FN - GM)}{WI(\lambda, \mu)}.$$

Hence, α_g is expressed by the coefficients of the first and the second fundamental forms in the same way as the geodesic torsion in the theory of surfaces in \mathbb{R}^3 .

We define asymptotic tangents and principal tangents as follows:

Definition 2.3. A tangent $g : X = \lambda z_u + \mu z_v$ is said to be *asymptotic* if it is self-conjugate, i.e. $\nu_g = 0$.

Definition 2.4. A tangent $g : X = \lambda z_u + \mu z_v$ is said to be *principal* if it is perpendicular to its conjugate, i.e. $\alpha_g = 0$.

The equation for the asymptotic tangents at a point $p \in M^2$ is

$$L\lambda^2 + 2M\lambda\mu + N\mu^2 = 0.$$

The equation for the principal tangents at a point $p \in M^2$ is

$$\begin{vmatrix} E & F \\ L & M \end{vmatrix} \lambda^2 + \begin{vmatrix} E & G \\ L & N \end{vmatrix} \lambda\mu + \begin{vmatrix} F & G \\ M & N \end{vmatrix} \mu^2 = 0.$$

A line $c : u = u(q), v = v(q); q \in J \subset \mathbb{R}$ on M^2 is said to be an *asymptotic line*, respectively a *principal line*, if its tangent at any point is asymptotic, respectively principal. The surface M^2 is parameterized by principal lines if and only if $F = 0, M = 0$.

3. WEINGARTEN MAP OF A SPACELIKE SURFACE IN \mathbb{R}_1^4

The second fundamental form II determines a map of Weingarten-type $\gamma : T_p M^2 \rightarrow T_p M^2$ at any point of M^2 in the standard way:

$$\begin{aligned} \gamma(z_u) &= \gamma_1^1 z_u + \gamma_1^2 z_v, \\ \gamma(z_v) &= \gamma_2^1 z_u + \gamma_2^2 z_v, \end{aligned}$$

where

$$\gamma_1^1 = \frac{FM - GL}{EG - F^2}, \quad \gamma_1^2 = \frac{FL - EM}{EG - F^2}, \quad \gamma_2^1 = \frac{FN - GM}{EG - F^2}, \quad \gamma_2^2 = \frac{FM - EN}{EG - F^2}.$$

The linear map γ is invariant under changes of the parameters of the surface and changes of the normal frame field. Hence the following statement holds.

Lemma 3.1. *The functions*

$$k := \det \gamma = \frac{LN - M^2}{EG - F^2}, \quad \varkappa := -\frac{1}{2} \operatorname{tr} \gamma = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

are invariants of the surface M^2 .

Now we shall prove

Proposition 3.2. *The function \varkappa is the curvature of the normal connection of M^2 .*

Proof: Let D be the normal connection of M^2 . For any tangent vector fields x, y and any normal vector field n we have the standard decomposition

$$\nabla'_x n = -A_n(x) + D_x n,$$

where $\langle A_n(x), y \rangle = \langle \sigma(x, y), n \rangle$.

The curvature tensor R^\perp of the normal connection D is given by $R^\perp(x, y)n = D_x D_y n - D_y D_x n - D_{[x, y]}n$. Then the curvature of the normal connection at a point $p \in M^2$ is defined by $\langle R^\perp(x, y)n_2, n_1 \rangle$, where $\{x, y, n_1, n_2\}$ is a right oriented orthonormal quadruple.

Without loss of generality we assume that $F = 0$ and denote the unit vector fields $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$. Then we have

$$\begin{aligned} \sigma(x, x) &= \frac{c_{11}^1}{E} n_1 - \frac{c_{11}^2}{E} n_2, \\ \sigma(x, y) &= \frac{c_{12}^1}{\sqrt{EG}} n_1 - \frac{c_{12}^2}{\sqrt{EG}} n_2, \\ \sigma(y, y) &= \frac{c_{22}^1}{G} n_1 - \frac{c_{22}^2}{G} n_2. \end{aligned}$$

Hence,

$$\begin{aligned} (3.1) \quad A_1(x) &= \frac{c_{11}^1}{E} x + \frac{c_{12}^1}{\sqrt{EG}} y, & A_2(x) &= \frac{c_{11}^2}{E} x + \frac{c_{12}^2}{\sqrt{EG}} y, \\ A_1(y) &= \frac{c_{12}^1}{\sqrt{EG}} x + \frac{c_{22}^1}{G} y, & A_2(y) &= \frac{c_{12}^2}{\sqrt{EG}} x + \frac{c_{22}^2}{G} y. \end{aligned}$$

Using (3.1) we calculate

$$\begin{aligned} (A_2 \circ A_1 - A_1 \circ A_2)(x) &= \left(\frac{c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1}{E \sqrt{EG}} + \frac{c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1}{G \sqrt{EG}} \right) y = \frac{EN + GL}{2EG} y; \\ (A_2 \circ A_1 - A_1 \circ A_2)(y) &= - \left(\frac{c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1}{E \sqrt{EG}} + \frac{c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1}{G \sqrt{EG}} \right) x = - \frac{EN + GL}{2EG} x. \end{aligned}$$

Hence,

$$\begin{aligned} (3.2) \quad (A_2 \circ A_1 - A_1 \circ A_2)(x) &= \varkappa y; \\ (A_2 \circ A_1 - A_1 \circ A_2)(y) &= -\varkappa x. \end{aligned}$$

Note that $A_2 \circ A_1 - A_1 \circ A_2$ is an invariant skew-symmetric operator in the tangent space, i.e. it does not depend on the choice of the orthonormal tangent frame field $\{x, y\}$.

Since the curvature tensor R' of the connection ∇' is zero, we have

$$\nabla'_x \nabla'_y n_1 - \nabla'_y \nabla'_x n_1 - \nabla'_{[x,y]} n_1 = 0.$$

Therefore the tangent component and the normal component of $R'(x, y)n_1$ are both zero. The normal component is $D_x D_y n_1 - D_y D_x n_1 - D_{[x,y]} n_1 - \sigma(x, A_1 y) + \sigma(y, A_1 x)$. Hence,

$$(3.3) \quad D_x D_y n_1 - D_y D_x n_1 - D_{[x,y]} n_1 = \sigma(x, A_1 y) - \sigma(y, A_1 x).$$

The left-hand side of (3.3) is $R^\perp(x, y)n_1$. Then

$$\langle R^\perp(x, y)n_1, n_2 \rangle = \langle \sigma(x, A_1 y), n_2 \rangle - \langle \sigma(y, A_1 x), n_2 \rangle = \langle (A_2 \circ A_1 - A_1 \circ A_2)(y), x \rangle.$$

Using (3.2) we obtain that $\langle R^\perp(x, y)n_1, n_2 \rangle = -\varkappa$. Since $\langle R^\perp(x, y)n_1, n_2 \rangle = -\langle R^\perp(x, y)n_2, n_1 \rangle$, we get

$$\langle R^\perp(x, y)n_2, n_1 \rangle = \varkappa.$$

The last equality implies that \varkappa is the curvature of the normal connection. \square

The characteristic equation of the Weingarten map γ is

$$\nu^2 + 2\varkappa\nu + k = 0.$$

Since γ is a symmetric linear operator, the following inequality holds:

$$\varkappa^2 - k \geq 0.$$

Moreover, the equality $\varkappa^2 - k = 0$ is equivalent to the conditions

$$L = \rho E, \quad M = \rho F, \quad N = \rho G, \quad \rho \in \mathbb{R}.$$

Obviously, the following equivalence at a point $p \in M^2$ holds:

$$L = M = N = 0 \quad \Longleftrightarrow \quad k = \varkappa = 0.$$

As in the theory of surfaces in \mathbb{R}^3 and \mathbb{R}^4 , the invariants k and \varkappa divide the points of M^2 into four types. A point $p \in M^2$ is said to be:

- flat*, if $k = \varkappa = 0$;
- elliptic*, if $k > 0$;
- parabolic*, if $k = 0$, $\varkappa \neq 0$;
- hyperbolic*, if $k < 0$.

Spacelike surfaces consisting of flat points will be considered in Section 4. Further in this section we shall consider surfaces free of flat points, i.e. $(L, M, N) \neq (0, 0, 0)$.

We note that a spacelike surface M^2 has two families of orthogonal asymptotic lines if and only if M^2 is of flat normal connection.

Let $H = \frac{1}{2}(\sigma(x, x) + \sigma(y, y))$ be the normal mean curvature vector field. We recall that a surface M^2 is said to be *minimal* if the mean curvature vector $H = 0$. The minimal surfaces are characterized in terms of the invariants k and \varkappa by the following

Proposition 3.3. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. Then M^2 is minimal if and only if*

$$\varkappa^2 - k = 0.$$

Proof: Without loss of generality we assume that $F = 0$ and denote the unit vector fields $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$. Then we have

$$\begin{aligned}\nabla'_x x &= \gamma_1 y + \frac{c_{11}^1}{E} n_1 - \frac{c_{11}^2}{E} n_2, \\ \nabla'_x y &= -\gamma_1 x + \frac{c_{12}^1}{\sqrt{EG}} n_1 - \frac{c_{12}^2}{\sqrt{EG}} n_2, \\ \nabla'_y x &= -\gamma_2 y + \frac{c_{12}^1}{\sqrt{EG}} n_1 - \frac{c_{12}^2}{\sqrt{EG}} n_2, \\ \nabla'_y y &= \gamma_2 x + \frac{c_{22}^1}{G} n_1 - \frac{c_{22}^2}{G} n_2.\end{aligned}$$

I. Let $H = \frac{1}{2}(\sigma(x, x) + \sigma(y, y)) = 0$. Then $c_{22}^1 = -\frac{G}{E} c_{11}^1$, $c_{22}^2 = -\frac{G}{E} c_{11}^2$, and hence

$$\Delta_2 = \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix} = 0, \quad \frac{\Delta_3}{G} = \frac{\Delta_1}{E}.$$

Therefore

$$L = \rho E, \quad M = \rho F, \quad N = \rho G,$$

where ρ is a function on M^2 . Hence $\varkappa^2 - k = 0$.

II. Let $\varkappa^2 - k = 0$. Then

$$L = \rho E, \quad M = \rho F, \quad N = \rho G; \quad \rho \neq 0.$$

The condition $F = 0$ implies that $M = 0$. Then $\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix} = 0$ and $c_{22}^1 = \tilde{\rho} c_{11}^1$, $c_{22}^2 = \tilde{\rho} c_{11}^2$.

Further, the equality $\frac{L}{E} = \frac{N}{G}$ implies that $\tilde{\rho} = -\frac{G}{E}$. Hence $\text{tr } \sigma = 0$, i.e. $H = 0$. \square

Let us note that the spacelike surfaces consisting of "umbilical" points in \mathbb{R}_1^4 are exactly the minimal surfaces.

We shall characterize the minimal surfaces and the surfaces with flat normal connection in terms of a geometric figure in the tangent space at any point of a spacelike surface.

The normal curvatures of the principal tangents are said to be *principal normal curvatures* of M^2 . If a point $p \in M^2$ is "non-umbilical", i.e. $\varkappa^2 - k > 0$, we can assume that (u, v) are principal parameters ($F = M = 0$). The principal normal curvatures are $\nu' = \frac{L}{E}$, $\nu'' = \frac{N}{G}$ and the invariants k and \varkappa of M^2 are expressed by the principal normal curvatures as follows:

$$(3.4) \quad k = \nu' \nu''; \quad \varkappa = \frac{\nu' + \nu''}{2}.$$

If $p \in M^2$ is an "umbilical" point, i.e. $\varkappa^2 - k = 0$, then all tangents at p are principal with one and the same normal curvature ν . Then formulas (3.4) are also valid under the assumption $\nu' = \nu'' = \nu$.

Similarly to the theory of surfaces in \mathbb{R}^3 and \mathbb{R}^4 , we consider the indicatrix χ in the tangent space $T_p M^2$ at an arbitrary point p of M^2 , defined by

$$\chi : \nu' X^2 + \nu'' Y^2 = \varepsilon, \quad \varepsilon = \pm 1.$$

If p is an elliptic point ($k > 0$), then the indicatrix χ is an ellipse. The axes of χ are collinear with the principal tangents at the point p , and the lengths of the axes are $\frac{2}{\sqrt{|\nu'|}}$

and $\frac{2}{\sqrt{|\nu''|}}$.

If p is a hyperbolic point ($k < 0$), then the indicatrix χ consists of two hyperbolas. For the sake of simplicity we say that χ is a hyperbola. The axes of χ are collinear with the principal tangents, and the lengths of the axes are $\frac{2}{\sqrt{|\nu'|}}$ and $\frac{2}{\sqrt{|\nu''|}}$.

If p is a parabolic point ($k = 0$), then the indicatrix χ consists of two straight lines parallel to the principal tangent with non-zero normal curvature.

The following statement holds:

Proposition 3.4. *Two tangents g_1 and g_2 are conjugate tangents of M^2 if and only if g_1 and g_2 are conjugate with respect to the indicatrix χ .*

Now we shall characterize the minimal surfaces and the surfaces with flat normal connection in terms of the tangent indicatrix of the surface.

Proposition 3.5. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. Then M^2 is minimal if and only if at each point of M^2 the tangent indicatrix χ is a circle.*

Proof: Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. From equalities (3.4) it follows that

$$\varkappa^2 - k = \left(\frac{\nu' - \nu''}{2} \right)^2.$$

Obviously $\varkappa^2 - k = 0$ if and only if $\nu' = \nu''$. Applying Proposition 3.3, we get that M^2 is minimal if and only if χ is a circle. \square

Proposition 3.6. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. Then M^2 is a surface with flat normal connection if and only if at each point of M^2 the tangent indicatrix χ is a rectangular hyperbola (a Lorentz circle).*

Proof: Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. From (3.4) it follows that $\varkappa = 0$ if and only if $\nu'' = -\nu'$.

If M^2 is a surface with flat normal connection, then $k < 0$, and hence χ is a hyperbola. From $\nu'' = -\nu'$ it follows that the semi-axes of χ are equal to $\frac{1}{\sqrt{|\nu'|}}$, i.e. χ is a rectangular hyperbola.

Conversely, if χ is a rectangular hyperbola, then $\nu'' = -\nu'$, which implies that M^2 is a surface with flat normal connection. \square

The minimal surfaces and the surfaces with flat normal connection can also be characterized in terms of the ellipse of normal curvature.

The notion of the ellipse of normal curvature of a surface in space forms was introduced by Moore and Wilson [11, 12]. The ellipse of normal curvature associated to the second fundamental form of a spacelike surface in \mathbb{R}_1^4 was first considered in [9]. The *ellipse of normal curvature* at a point p of a surface M^2 is the ellipse in the normal space at the point p given by $\{\sigma(x, x) : x \in T_p M^2, \langle x, x \rangle = 1\}$. Let $\{x, y\}$ be an orthonormal base of the tangent space $T_p M^2$ at p . Then, for any $v = \cos \psi x + \sin \psi y$, we have

$$(3.5) \quad \sigma(v, v) = H + \cos 2\psi \frac{\sigma(x, x) - \sigma(y, y)}{2} + \sin 2\psi \sigma(x, y),$$

where H is the mean curvature vector of M^2 at p . So, when v goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice around the ellipse centered at H . The vectors $\frac{\sigma(x, x) - \sigma(y, y)}{2}$ and $\sigma(x, y)$ determine conjugate directions of the ellipse.

Obviously, M^2 is minimal if and only if for each point $p \in M^2$ the ellipse of curvature is centered at p . We shall give a characterization of the surfaces with flat normal connection in terms of the ellipse of normal curvature.

Lemma 3.7. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points and x, y be principal tangents. Then M^2 is a surface with flat normal connection if and only if $\sigma(x, x) = \sigma(y, y)$.*

Proof: Let M^2 be a surface in \mathbb{R}_1^4 free of flat points, and parameterized by principal parameters, i.e. $F = M = 0$. Then $\varkappa = \frac{EN + GL}{2EG}$. Denote $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$. Since $M = 0$, we

have $\begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix} = 0$ and hence $c_{22}^1 = \rho c_{11}^1$, $c_{22}^2 = \rho c_{11}^2$. Then $\Delta_3 = -\rho \Delta_1$ and $N = -\rho L$.

I. Let M^2 be of flat normal connection, i.e. $\varkappa = 0$. Then, $N = -\frac{G}{E}L$, and hence, $\rho = \frac{G}{E}$, which implies that $c_{22}^1 = \frac{G}{E}c_{11}^1$, $c_{22}^2 = \frac{G}{E}c_{11}^2$. Consequently,

$$\sigma(y, y) = \frac{c_{22}^1}{G}n_1 - \frac{c_{22}^2}{G}n_2 = \frac{c_{11}^1}{E}n_1 - \frac{c_{11}^2}{E}n_2 = \sigma(x, x).$$

II. Let $\sigma(x, x) = \sigma(y, y)$. Then $\frac{c_{22}^1}{G} = \frac{c_{11}^1}{E}$; $\frac{c_{22}^2}{G} = \frac{c_{11}^2}{E}$. Using that $c_{22}^1 = \rho c_{11}^1$, $c_{22}^2 = \rho c_{11}^2$, we get $\rho = \frac{G}{E}$, and hence $N = -\frac{G}{E}L$. The last equality implies $\varkappa = 0$, i.e. M^2 is a surface of flat normal connection. \square

Proposition 3.8. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 free of flat points. Then M^2 is a surface of flat normal connection if and only if for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with the mean curvature vector field.*

Proof: I. Let M^2 be a surface of flat normal connection. According to Lemma 3.7 we have $\sigma(x, x) - \sigma(y, y) = 0$. Then for any $v = \cos \psi x + \sin \psi y$, we get $\sigma(v, v) = H + \sin 2\psi \sigma(x, y)$. Hence, when v goes once around the unit tangent circle, the vector $\sigma(v, v)$ goes twice along the line segment collinear with $\sigma(x, y)$ and centered at H . If we assume that $\sigma(x, y)$ is collinear with $H = \sigma(x, x)$, we get $L = M = N = 0$, which contradicts the condition that M^2 is free of flat points.

II. Let M^2 be a surface in \mathbb{R}_1^4 free of flat points such that for each point $p \in M^2$ the ellipse of normal curvature is a line segment, which is not collinear with H . Without loss of generality we assume that M^2 is parameterized by principal parameters, and hence $c_{22}^1 = \rho c_{11}^1$, $c_{22}^2 = \rho c_{11}^2$. Then

$$(3.6) \quad \begin{aligned} \sigma(x, x) - \sigma(y, y) &= (1 - \rho)(c_{11}^1 n_1 + c_{11}^2 n_2); \\ \sigma(x, y) &= c_{12}^1 n_1 + c_{12}^2 n_2. \end{aligned}$$

Since the ellipse of normal curvature is a line segment, having in mind (3.5), we get one of the following possibilities:

(a) $\sigma(x, x) - \sigma(y, y)$ is collinear with $\sigma(x, y)$. In this case from (3.6) we get $c_{12}^1 = \tilde{\rho} c_{11}^1$; $c_{12}^2 = \tilde{\rho} c_{11}^2$, which implies $L = M = N = 0$, a contradiction.

(b) $\sigma(x, y) = 0$, which implies $c_{12}^1 = c_{12}^2 = 0$, and hence $L = M = N = 0$, a contradiction.

(c) $\sigma(x, x) - \sigma(y, y) = 0$. i.e. $\sigma(x, x) = \sigma(y, y)$. Applying Lemma 3.7, we get that M^2 is a surface of flat normal connection. \square

4. SPACELIKE SURFACES CONSISTING OF FLAT POINTS

In this section we consider spacelike surfaces consisting of flat points, i.e. surfaces satisfying the conditions

$$k(u, v) = 0, \quad \varkappa(u, v) = 0, \quad (u, v) \in \mathcal{D},$$

or equivalently $L(u, v) = 0, M(u, v) = 0, N(u, v) = 0, (u, v) \in \mathcal{D}$.

We shall give a local geometric description of those such surfaces whose mean curvature vector H at any point is:

- (1) a non-zero spacelike vector, i.e. $\langle H, H \rangle > 0$, or
- (2) a timelike vector, i.e. $\langle H, H \rangle < 0$.

Theorem 4.1. *Let M^2 be a spacelike surface in \mathbb{R}_1^4 consisting of flat points and the mean curvature vector at any point is a non-zero spacelike vector or timelike vector. Then either M^2 lies in a hyperplane of \mathbb{R}_1^4 or M^2 is part of a developable ruled surface in \mathbb{R}_1^4 .*

Proof: Let $M^2 : z = z(u, v), (u, v) \in \mathcal{D} (\mathcal{D} \subset \mathbb{R}^2)$ be a spacelike surface in \mathbb{R}_1^4 whose mean curvature vector at any point is a non-zero spacelike vector or timelike vector and $L(u, v) = M(u, v) = N(u, v) = 0, (u, v) \in \mathcal{D}$. For the sake of simplicity, we assume that the parametrization of M^2 is orthogonal, i.e. $F = 0$. Denote the unit vector fields $x = \frac{z_u}{\sqrt{E}}, y = \frac{z_v}{\sqrt{G}}$. The conditions $L = M = N = 0$ imply that

$$\text{rank} \begin{pmatrix} c_{11}^1 & c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 & c_{22}^2 \end{pmatrix} = 1$$

and the vectors $\sigma(x, x), \sigma(x, y), \sigma(y, y)$ are collinear. Let n be a unit normal vector field of M^2 , which is collinear with $\sigma(x, x), \sigma(x, y)$, and $\sigma(y, y)$. Hence, n is collinear with the mean curvature vector field H . We have the following possibilities:

- (1) n is spacelike, i.e. $\langle n, n \rangle = 1$.
- (2) n is timelike, i.e. $\langle n, n \rangle = -1$.

First we shall consider the case $\langle n, n \rangle = 1$. Denote by l the unit normal vector field such that $\{x, y, n, l\}$ is a positively oriented orthonormal frame field in \mathbb{R}_1^4 (hence $\langle l, l \rangle = -1$). It is clear that the normal vector fields n, l are determined up to a sign. Then we have the following derivative formulas of M^2 :

$$(4.1) \quad \begin{aligned} \nabla'_x x &= \gamma_1 y + \nu_1 n, & \nabla'_x n &= -\nu_1 x - \lambda y - \beta_1 l, \\ \nabla'_x y &= -\gamma_1 x + \lambda n, & \nabla'_y n &= -\lambda x - \nu_2 y - \beta_2 l, \\ \nabla'_y x &= -\gamma_2 y + \lambda n, & \nabla'_x l &= -\beta_1 n, \\ \nabla'_y y &= \gamma_2 x + \nu_2 n, & \nabla'_y l &= -\beta_2 n, \end{aligned}$$

where $\nu_1, \nu_2, \lambda, \beta_1, \beta_2, \gamma_1, \gamma_2$ are functions on M^2 .

The mean curvature vector field is $H = \frac{\nu_1 + \nu_2}{2} n$. The Gauss curvature K of M^2 is expressed by

$$(4.2) \quad K = \nu_1 \nu_2 - \lambda^2.$$

Since the curvature tensor R' of the connection ∇' is zero, then the equality $R'(x, y, l) = 0$ together with (4.1) imply that

$$(4.3) \quad \begin{aligned} \nu_1 \beta_2 - \lambda \beta_1 &= 0; \\ -\lambda \beta_2 + \nu_2 \beta_1 &= 0. \end{aligned}$$

We have two subcases:

(a) $\beta_1 = \beta_2 = 0$ for all $(u, v) \in \mathcal{D}$. Then from equalities (4.1) it follows that $\nabla'_x l = 0$; $\nabla'_y l = 0$, and hence $l = \text{const}$. Consequently, M^2 lies in a hyperplane \mathbb{E}^3 of \mathbb{R}_1^4 orthogonal to l .

(b) There exists a point $(u_0, v_0) \in \mathcal{D}$ such that $\beta_1^2(u_0, v_0) + \beta_2^2(u_0, v_0) \neq 0$. Hence, there exists a neighborhood $\mathcal{D}_0 \subset \mathcal{D}$ such that $(\beta_1^2 + \beta_2^2)|_{\mathcal{D}_0} \neq 0$. Then, equalities (4.3) imply that $\nu_1 \nu_2 - \lambda^2 = 0$ for $(u, v) \in \mathcal{D}_0$. If $\nu_1 = \nu_2 = 0$, then $H = 0$, which contradicts the assumption that $\langle H, H \rangle > 0$. So we assume that there exists a neighborhood $\tilde{\mathcal{D}} \subset \mathcal{D}_0$ such that $\nu_2|_{\tilde{\mathcal{D}}} \neq 0$ (or $\nu_1|_{\tilde{\mathcal{D}}} \neq 0$). We consider the surface $\tilde{M}^2 = M^2|_{\tilde{\mathcal{D}}}$, which is a surface with zero Gauss curvature in view of (4.2).

Let $\{\bar{x}, \bar{y}\}$ be the orthonormal tangent frame field of \tilde{M}^2 , defined by

$$\begin{aligned} \bar{x} &= \cos \varphi x + \sin \varphi y; \\ \bar{y} &= -\sin \varphi x + \cos \varphi y, \end{aligned}$$

where $\tan \varphi = -\frac{\lambda}{\nu_2}$. Then $\sigma(\bar{x}, \bar{x}) = 0$, $\sigma(\bar{x}, \bar{y}) = 0$. So, formulas (4.1) take the form

$$\begin{aligned} \nabla'_{\bar{x}} \bar{x} &= \bar{\gamma}_1 \bar{y}, & \nabla'_{\bar{x}} n &= -\bar{\beta}_1 l, \\ \nabla'_{\bar{x}} \bar{y} &= -\bar{\gamma}_1 \bar{x}, & \nabla'_{\bar{y}} n &= -\bar{\nu}_2 \bar{y} - \bar{\beta}_2 l, \\ \nabla'_{\bar{y}} \bar{x} &= -\bar{\gamma}_2 \bar{y}, & \nabla'_{\bar{x}} l &= -\bar{\beta}_1 n, \\ \nabla'_{\bar{y}} \bar{y} &= \bar{\gamma}_2 \bar{x} + \bar{\nu}_2 n, & \nabla'_{\bar{y}} l &= -\bar{\beta}_2 n, \end{aligned}$$

where $\bar{\nu}_2 \neq 0$.

Now the equalities $R'(\bar{x}, \bar{y}, n) = 0$ and $R'(\bar{x}, \bar{y}, l) = 0$ imply that

$$\bar{\gamma}_1 = 0, \quad \bar{\beta}_1 = 0.$$

Hence,

$$\begin{aligned} \nabla'_{\bar{x}} \bar{x} &= 0, & \nabla'_{\bar{x}} n &= 0, \\ \nabla'_{\bar{x}} \bar{y} &= 0, & \nabla'_{\bar{x}} l &= 0. \end{aligned}$$

Let $p = z(\bar{u}_0, \bar{v}_0)$, $(\bar{u}_0, \bar{v}_0) \in \tilde{\mathcal{D}}$ be an arbitrary point of \tilde{M}^2 and $c_1 : z(\bar{u}) = z(\bar{u}, \bar{v}_0)$ be the integral curve of the vector field \bar{x} , passing through p . It follows from $\nabla'_{\bar{x}} \bar{x} = 0$ that c_1 is contained in a straight line. Hence, \tilde{M}^2 lies on a one-parameter family of straight lines, i.e. \tilde{M}^2 lies on a ruled surface. Moreover, since $\nabla'_{\bar{x}} n = 0$ and $\nabla'_{\bar{x}} l = 0$ then the normal space $\text{span}\{n, l\}$ of \tilde{M}^2 is constant at the points of c_1 and hence, the tangent space $\text{span}\{\bar{x}, \bar{y}\}$ of \tilde{M}^2 at the points of c_1 is one and the same. Consequently, \tilde{M}^2 is part of a developable surface.

Now we shall consider the case $\langle n, n \rangle = -1$. Denote by b the unit normal vector field such that $\{x, y, b, n\}$ is a positively oriented orthonormal frame field in \mathbb{R}_1^4 (hence $\langle b, b \rangle = 1$).

The normal vector fields b, n are determined up to a sign. In this case we have the following derivative formulas of M^2 :

$$(4.4) \quad \begin{aligned} \nabla'_x x &= \gamma_1 y - \nu_1 n, & \nabla'_x b &= -\beta_1 n, \\ \nabla'_x y &= -\gamma_1 x - \lambda n, & \nabla'_y b &= -\beta_2 n, \\ \nabla'_y x &= -\gamma_2 y - \lambda n, & \nabla'_x n &= -\nu_1 x - \lambda y - \beta_1 b, \\ \nabla'_y y &= \gamma_2 x - \nu_2 n, & \nabla'_y n &= -\lambda x - \nu_2 y - \beta_2 b, \end{aligned}$$

where $\nu_1, \nu_2, \lambda, \beta_1, \beta_2, \gamma_1, \gamma_2$ are functions on M^2 .

The mean curvature vector field is $H = -\frac{\nu_1 + \nu_2}{2} n$. The Gauss curvature K of M^2 is $K = -\nu_1 \nu_2 + \lambda^2$. As in the previous case, using that $R'(x, y, b) = 0$ from equalities (4.4) we obtain equalities (4.3) which imply that there are two subcases:

(a) $\beta_1 = \beta_2 = 0$ for all $(u, v) \in \mathcal{D}$. Then $\nabla'_x b = 0$; $\nabla'_y b = 0$, and hence $b = \text{const}$. Consequently, M^2 lies in a hyperplane \mathbb{E}^3 of \mathbb{R}_1^4 orthogonal to b .

(b) There exists a point $(u_0, v_0) \in \mathcal{D}$ such that $\beta_1^2(u_0, v_0) + \beta_2^2(u_0, v_0) \neq 0$. In the same way as in the first case we obtain that in a neighborhood $\tilde{\mathcal{D}} \subset \mathcal{D}$ the surface $\tilde{M}^2 = M^2|_{\tilde{\mathcal{D}}}$ is part of a developable surface. \square

5. SPACELIKE SURFACES WHOSE MEAN CURVATURE VECTOR AT ANY POINT IS A NON-ZERO SPACELIKE VECTOR

Let M^2 be a spacelike surface parameterized by principal lines and $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$. The equality $M = 0$ implies that the normal vector fields $\sigma(x, x)$ and $\sigma(y, y)$ are collinear. Hence, there exists a geometrically determined normal frame field n , such that $\sigma(x, x)$ and $\sigma(y, y)$ are collinear with n . Then we have the following formulas:

$$\begin{aligned} \sigma(x, x) &= \nu_1 n, \\ \sigma(y, y) &= \nu_2 n, \end{aligned}$$

where ν_1, ν_2 are invariant functions. The mean curvature vector field is expressed as follows:

$$H = \frac{\nu_1 + \nu_2}{2} n.$$

Let M^2 be free of minimal points, i.e. $H \neq 0$ at each point of M^2 . We have the following possibilities for the mean curvature vector field:

- (1) H is *spacelike*, i.e. $\langle H, H \rangle > 0$.
- (2) H is *timelike*, i.e. $\langle H, H \rangle < 0$.
- (3) H is *lightlike*, i.e. $\langle H, H \rangle = 0$.

In this section we shall consider spacelike surfaces whose mean curvature vector at any point is a non-zero spacelike vector. Let x, y be the principal tangent vector fields. We denote by b the unit normal vector field $b = \frac{H}{\sqrt{\langle H, H \rangle}}$. We have $\langle b, b \rangle = 1$ and b is collinear with $\sigma(x, x)$ and $\sigma(y, y)$. Denote by l the unit normal vector field such that $\{x, y, b, l\}$ is a positively oriented orthonormal frame field in \mathbb{R}_1^4 (hence $\langle l, l \rangle = -1$). Thus we obtain a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point $p \in M^2$. With

respect to the frame field $\{x, y, b, l\}$ we have the following formulas:

$$(5.1) \quad \begin{aligned} \sigma(x, x) &= \nu_1 b; \\ \sigma(x, y) &= \lambda b - \mu l; \\ \sigma(y, y) &= \nu_2 b, \end{aligned}$$

where $\nu_1, \nu_2, \lambda, \mu$ are invariant functions, $\nu_1 = \langle \sigma(x, x), b \rangle$, $\nu_2 = \langle \sigma(y, y), b \rangle$, $\lambda = \langle \sigma(x, y), b \rangle$, $\mu = \langle \sigma(x, y), l \rangle$.

The invariants k , \varkappa , and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows:

$$(5.2) \quad k = -4\nu_1 \nu_2 \mu^2, \quad \varkappa = (\nu_1 - \nu_2)\mu, \quad K = \nu_1 \nu_2 - \lambda^2 + \mu^2.$$

Since $\varkappa^2 - k > 0$, equalities (5.2) imply that $\mu \neq 0$.

The normal mean curvature vector field of M^2 is $H = \frac{\nu_1 + \nu_2}{2} b$. Taking into account (5.2) we obtain that the length $\|H\|$ of the mean curvature vector field is given by the formula

$$\|H\| = \frac{\sqrt{\varkappa^2 - k}}{2|\mu|},$$

which shows that $|\mu|$ is expressed by the invariants k , \varkappa and the mean curvature function.

Now we shall discuss the geometric meaning of the invariant λ . Let M be an n -dimensional submanifold of $(n+m)$ -dimensional Riemannian manifold \widetilde{M} and ξ be a normal vector field of M . In [1] B.-Y. Chen defined the *allied vector field* $a(\xi)$ of ξ by the formula

$$a(\xi) = \frac{\|\xi\|}{n} \sum_{k=2}^m \{\text{tr}(A_1 A_k)\} \xi_k,$$

where $\{\xi_1 = \frac{\xi}{\|\xi\|}, \xi_2, \dots, \xi_m\}$ is an orthonormal base of the normal space of M , and $A_i = A_{\xi_i}$, $i = 1, \dots, m$ is the shape operator with respect to ξ_i . In particular, the allied vector field $a(H)$ of the mean curvature vector field H is a well-defined normal vector field which is orthogonal to H . It is called the *allied mean curvature vector field* of M in \widetilde{M} . B.-Y. Chen defined the \mathcal{A} -submanifolds to be those submanifolds of \widetilde{M} for which $a(H)$ vanishes identically [1]. In [6], [7] the \mathcal{A} -submanifolds are called *Chen submanifolds*. It is easy to see that minimal submanifolds, pseudo-umbilical submanifolds and hypersurfaces are Chen submanifolds. These Chen submanifolds are said to be trivial \mathcal{A} -submanifolds. Now let M^2 be a spacelike surface in \mathbb{R}_1^4 with spacelike mean curvature vector field. Applying the definition of the allied mean curvature vector field from equalities (5.1) we get

$$a(H) = \frac{\nu_1 + \nu_2}{2} \lambda \mu l = \frac{\sqrt{\varkappa^2 - k}}{2} \lambda l.$$

Hence, if M^2 is free of minimal points, then $a(H) = 0$ if and only if $\lambda = 0$. This gives the geometric meaning of the invariant λ . It is clear that M^2 is a non-trivial Chen surface if and only if the invariant λ is zero.

With respect to the geometrically determined orthonormal frame field $\{x, y, b, l\}$ we have the following Frenet-type derivative formulas of M^2 :

$$(5.3) \quad \begin{aligned} \nabla'_x x &= \gamma_1 y + \nu_1 b; & \nabla'_x b &= -\nu_1 x - \lambda y - \beta_1 l; \\ \nabla'_x y &= -\gamma_1 x + \lambda b - \mu l; & \nabla'_y b &= -\lambda x - \nu_2 y - \beta_2 l; \\ \nabla'_y x &= -\gamma_2 y + \lambda b - \mu l; & \nabla'_x l &= -\mu y - \beta_1 b; \\ \nabla'_y y &= \gamma_2 x + \nu_2 b; & \nabla'_y l &= -\mu x - \beta_2 b, \end{aligned}$$

where $\gamma_1 = -y(\ln \sqrt{E})$, $\gamma_2 = -x(\ln \sqrt{G})$, $\beta_1 = \langle \nabla'_x b, l \rangle$, $\beta_2 = \langle \nabla'_y b, l \rangle$.

Using that $R'(x, y, x) = 0$, $R'(x, y, y) = 0$, and $R'(x, y, b) = 0$, from (5.3) we get the following integrability conditions:

$$\begin{aligned} \nu_1 \nu_2 - \lambda^2 + \mu^2 &= x(\gamma_2) + y(\gamma_1) - ((\gamma_1)^2 + (\gamma_2)^2); \\ 2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1 &= x(\mu); \\ 2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1 &= y(\mu); \\ 2\lambda \gamma_2 - \mu \beta_1 - (\nu_1 - \nu_2) \gamma_1 &= x(\lambda) - y(\nu_1); \\ 2\lambda \gamma_1 - \mu \beta_2 + (\nu_1 - \nu_2) \gamma_2 &= -x(\nu_2) + y(\lambda); \\ \gamma_1 \beta_1 - \gamma_2 \beta_2 + (\nu_1 - \nu_2) \mu &= -x(\beta_2) + y(\beta_1). \end{aligned}$$

Having in mind that $x = \frac{z_u}{\sqrt{E}}$, $y = \frac{z_v}{\sqrt{G}}$, we can rewrite the above equalities in the following way:

$$\begin{aligned} \nu_1 \nu_2 - \lambda^2 + \mu^2 &= \frac{1}{\sqrt{E}} (\gamma_2)_u + \frac{1}{\sqrt{G}} (\gamma_1)_v - ((\gamma_1)^2 + (\gamma_2)^2); \\ 2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1 &= \frac{1}{\sqrt{E}} \mu_u; \\ 2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1 &= \frac{1}{\sqrt{G}} \mu_v; \\ 2\lambda \gamma_2 - \mu \beta_1 - (\nu_1 - \nu_2) \gamma_1 &= \frac{1}{\sqrt{E}} \lambda_u - \frac{1}{\sqrt{G}} (\nu_1)_v; \\ 2\lambda \gamma_1 - \mu \beta_2 + (\nu_1 - \nu_2) \gamma_2 &= -\frac{1}{\sqrt{E}} (\nu_2)_u + \frac{1}{\sqrt{G}} \lambda_v; \\ \gamma_1 \beta_1 - \gamma_2 \beta_2 + (\nu_1 - \nu_2) \mu &= -\frac{1}{\sqrt{E}} (\beta_2)_u + \frac{1}{\sqrt{G}} (\beta_1)_v. \end{aligned}$$

The condition $\mu_u \mu_v \neq 0$ is equivalent to $(2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1)(2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1) \neq 0$. So, if $\mu_u \mu_v \neq 0$, then

$$\sqrt{E} = \frac{\mu_u}{2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1}; \quad \sqrt{G} = \frac{\mu_v}{2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1}.$$

We shall prove the following Bonnet-type theorem for spacelike surfaces in \mathbb{R}_1^4 whose mean curvature vector at any point is a non-zero spacelike vector.

Theorem 5.1. *Let $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ be smooth functions, defined in a domain \mathcal{D} , $\mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned}
 & \frac{\mu_u}{2\mu\gamma_2 + \nu_1\beta_2 - \lambda\beta_1} > 0; \\
 & \frac{\mu_v}{2\mu\gamma_1 - \lambda\beta_2 + \nu_2\beta_1} > 0; \\
 & -\gamma_1\sqrt{E}\sqrt{G} = (\sqrt{E})_v; \\
 & -\gamma_2\sqrt{E}\sqrt{G} = (\sqrt{G})_u; \\
 (5.4) \quad & \nu_1\nu_2 - \lambda^2 + \mu^2 = \frac{1}{\sqrt{E}}(\gamma_2)_u + \frac{1}{\sqrt{G}}(\gamma_1)_v - ((\gamma_1)^2 + (\gamma_2)^2); \\
 & 2\lambda\gamma_2 - \mu\beta_1 - (\nu_1 - \nu_2)\gamma_1 = \frac{1}{\sqrt{E}}\lambda_u - \frac{1}{\sqrt{G}}(\nu_1)_v; \\
 & 2\lambda\gamma_1 - \mu\beta_2 + (\nu_1 - \nu_2)\gamma_2 = -\frac{1}{\sqrt{E}}(\nu_2)_u + \frac{1}{\sqrt{G}}\lambda_v; \\
 & \gamma_1\beta_1 - \gamma_2\beta_2 + (\nu_1 - \nu_2)\mu = -\frac{1}{\sqrt{E}}(\beta_2)_u + \frac{1}{\sqrt{G}}(\beta_1)_v,
 \end{aligned}$$

where $\sqrt{E} = \frac{\mu_u}{2\mu\gamma_2 + \nu_1\beta_2 - \lambda\beta_1}$, $\sqrt{G} = \frac{\mu_v}{2\mu\gamma_1 - \lambda\beta_2 + \nu_2\beta_1}$. Let $\{x_0, y_0, b_0, l_0\}$ be an orthonormal frame at a point $p_0 \in \mathbb{R}_1^4$. Then there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique spacelike surface $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}_0$, whose mean curvature vector at any point is a non-zero spacelike vector. Moreover, M^2 passes through p_0 , the functions $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ are the geometric functions of M^2 and $\{x_0, y_0, b_0, l_0\}$ is the geometric frame of M^2 at the point p_0 .

Proof: We consider the following system of partial differential equations for the unknown vector functions $x = x(u, v)$, $y = y(u, v)$, $b = b(u, v)$, $l = l(u, v)$ in \mathbb{R}_1^4 :

$$\begin{aligned}
 (5.5) \quad & \begin{aligned} x_u &= \sqrt{E}\gamma_1 y + \sqrt{E}\nu_1 b & x_v &= -\sqrt{G}\gamma_2 y + \sqrt{G}\lambda b - \sqrt{G}\mu l \\ y_u &= -\sqrt{E}\gamma_1 x + \sqrt{E}\lambda b - \sqrt{E}\mu l & y_v &= \sqrt{G}\gamma_2 x + \sqrt{G}\nu_2 b \\ b_u &= -\sqrt{E}\nu_1 x - \sqrt{E}\lambda y - \sqrt{E}\beta_1 l & b_v &= -\sqrt{G}\lambda x - \sqrt{G}\nu_2 y - \sqrt{G}\beta_2 l \\ l_u &= -\sqrt{E}\mu y - \sqrt{E}\beta_1 b & l_v &= -\sqrt{G}\mu x - \sqrt{G}\beta_2 b \end{aligned}
 \end{aligned}$$

We denote

$$Z = \begin{pmatrix} x \\ y \\ b \\ l \end{pmatrix}; \quad A = \sqrt{E} \begin{pmatrix} 0 & \gamma_1 & \nu_1 & 0 \\ -\gamma_1 & 0 & \lambda & -\mu \\ -\nu_1 & -\lambda & 0 & -\beta_1 \\ 0 & -\mu & -\beta_1 & 0 \end{pmatrix}; \quad B = \sqrt{G} \begin{pmatrix} 0 & -\gamma_2 & \lambda & -\mu \\ \gamma_2 & 0 & \nu_2 & 0 \\ -\lambda & -\nu_2 & 0 & -\beta_2 \\ -\mu & 0 & -\beta_2 & 0 \end{pmatrix}.$$

Then system (5.5) can be rewritten in the form:

$$\begin{aligned}
 (5.6) \quad & Z_u = A Z, \\
 & Z_v = B Z.
 \end{aligned}$$

The integrability conditions of (5.6) are

$$Z_{uv} = Z_{vu},$$

i.e.

$$(5.7) \quad \frac{\partial a_i^k}{\partial v} - \frac{\partial b_i^k}{\partial u} + \sum_{j=1}^4 (a_i^j b_j^k - b_i^j a_j^k) = 0, \quad i, k = 1, \dots, 4,$$

where a_i^j and b_i^j are the elements of the matrices A and B . Using (5.4) we obtain that equalities (5.7) are fulfilled. Hence, there exists a subset $\mathcal{D}_1 \subset \mathcal{D}$ and unique vector functions $x = x(u, v)$, $y = y(u, v)$, $b = b(u, v)$, $l = l(u, v)$, $(u, v) \in \mathcal{D}_1$, which satisfy system (5.5) and the conditions

$$x(u_0, v_0) = x_0, \quad y(u_0, v_0) = y_0, \quad b(u_0, v_0) = b_0, \quad l(u_0, v_0) = l_0.$$

We shall prove that $x(u, v)$, $y(u, v)$, $b(u, v)$, $l(u, v)$ form an orthonormal frame in \mathbb{R}_1^4 for each $(u, v) \in \mathcal{D}_1$. Let us consider the following functions:

$$\begin{aligned} \varphi_1 &= \langle x, x \rangle - 1; & \varphi_5 &= \langle x, y \rangle; & \varphi_8 &= \langle y, b \rangle; \\ \varphi_2 &= \langle y, y \rangle - 1; & \varphi_6 &= \langle x, b \rangle; & \varphi_9 &= \langle y, l \rangle; \\ \varphi_3 &= \langle b, b \rangle - 1; & \varphi_7 &= \langle x, l \rangle; & \varphi_{10} &= \langle b, l \rangle; \\ \varphi_4 &= \langle l, l \rangle + 1; \end{aligned}$$

defined for each $(u, v) \in \mathcal{D}_1$. Using that $x(u, v)$, $y(u, v)$, $b(u, v)$, $l(u, v)$ satisfy (5.5), we obtain the system

$$(5.8) \quad \begin{aligned} \frac{\partial \varphi_i}{\partial u} &= \alpha_i^j \varphi_j, \\ \frac{\partial \varphi_i}{\partial v} &= \beta_i^j \varphi_j; \end{aligned} \quad i = 1, \dots, 10,$$

where α_i^j, β_i^j , $i, j = 1, \dots, 10$ are functions of $(u, v) \in \mathcal{D}_1$. System (5.8) is a linear system of partial differential equations for the functions $\varphi_i(u, v)$, $i = 1, \dots, 10$, $(u, v) \in \mathcal{D}_1$, satisfying $\varphi_i(u_0, v_0) = 0$, $i = 1, \dots, 10$. Hence, $\varphi_i(u, v) = 0$, $i = 1, \dots, 10$ for each $(u, v) \in \mathcal{D}_1$. Consequently, the vector functions $x(u, v)$, $y(u, v)$, $b(u, v)$, $l(u, v)$ form an orthonormal frame in \mathbb{R}_1^4 for each $(u, v) \in \mathcal{D}_1$.

Now, let us consider the system

$$(5.9) \quad \begin{aligned} z_u &= \sqrt{E} x \\ z_v &= \sqrt{G} y \end{aligned}$$

of partial differential equations for the vector function $z(u, v)$. Using (5.4) and (5.5) we get that the integrability conditions $z_{uv} = z_{vu}$ of system (5.9) are fulfilled. Hence, there exists a subset $\mathcal{D}_0 \subset \mathcal{D}_1$ and a unique vector function $z = z(u, v)$, defined for $(u, v) \in \mathcal{D}_0$ and satisfying $z(u_0, v_0) = p_0$.

Consequently, the surface $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}_0$ satisfies the assertion of the theorem. \square

6. SPACELIKE SURFACES WITH TIMELIKE MEAN CURVATURE VECTOR FIELD

Now we shall consider spacelike surfaces with timelike mean curvature vector field, i.e. $H \neq 0$, $\langle H, H \rangle < 0$. Let x, y be the principal tangent vector fields. We denote by l the unit normal vector field $l = -\frac{H}{\sqrt{-\langle H, H \rangle}}$. We have $\langle l, l \rangle = -1$ and l is collinear with $\sigma(x, x)$ and $\sigma(y, y)$. Denote by b the unit normal vector field ($\langle b, b \rangle = 1$) such that the quadruple $\{x, y, b, l\}$ is a positively oriented orthonormal frame field in \mathbb{R}_1^4 . Thus we obtain

a geometrically determined orthonormal frame field $\{x, y, b, l\}$ at each point $p \in M^2$. With respect to the frame field $\{x, y, b, l\}$ we have the following formulas:

$$\begin{aligned}\sigma(x, x) &= -\nu_1 l; \\ \sigma(x, y) &= \mu b - \lambda l; \\ \sigma(y, y) &= -\nu_2 l,\end{aligned}$$

where $\nu_1, \nu_2, \lambda, \mu$ are invariant functions, $\nu_1 = \langle \sigma(x, x), l \rangle$, $\nu_2 = \langle \sigma(y, y), l \rangle$, $\lambda = \langle \sigma(x, y), l \rangle$, $\mu = \langle \sigma(x, y), b \rangle$.

The invariants k, \varkappa , and the Gauss curvature K of M^2 are expressed by the functions $\nu_1, \nu_2, \lambda, \mu$ as follows:

$$(6.1) \quad k = -4\nu_1 \nu_2 \mu^2, \quad \varkappa = (\nu_1 - \nu_2)\mu, \quad K = -\nu_1 \nu_2 + \lambda^2 - \mu^2.$$

Since $\varkappa^2 - k > 0$, equalities (6.1) imply that $\mu \neq 0$. The normal mean curvature vector field of M^2 is $H = -\frac{\nu_1 + \nu_2}{2} l$. The allied mean curvature vector field is

$$a(H) = \frac{\nu_1 + \nu_2}{2} \lambda \mu b.$$

As in the previous section we see that M^2 is a non-trivial Chen surface if and only if the invariant λ is zero.

Now the Frenet-type derivative formulas of M^2 are:

$$(6.2) \quad \begin{aligned}\nabla'_x x &= \gamma_1 y - \nu_1 l; & \nabla'_x b &= -\mu y - \beta_1 l; \\ \nabla'_x y &= -\gamma_1 x + \mu b - \lambda l; & \nabla'_y b &= -\mu x - \beta_2 l; \\ \nabla'_y x &= -\gamma_2 y + \mu b - \lambda l; & \nabla'_x l &= -\nu_1 x - \lambda y - \beta_1 b; \\ \nabla'_y y &= \gamma_2 x - \nu_2 l; & \nabla'_y l &= -\lambda x - \nu_2 y - \beta_2 b,\end{aligned}$$

where $\gamma_1 = -y(\ln \sqrt{E})$, $\gamma_2 = -x(\ln \sqrt{G})$, $\beta_1 = \langle \nabla'_x b, l \rangle$, $\beta_2 = \langle \nabla'_y b, l \rangle$.

Using that $R'(x, y, x) = 0$, $R'(x, y, y) = 0$, and $R'(x, y, b) = 0$, from (6.2) we get the following integrability conditions:

$$\begin{aligned}-\nu_1 \nu_2 + \lambda^2 - \mu^2 &= x(\gamma_2) + y(\gamma_1) - ((\gamma_1)^2 + (\gamma_2)^2); \\ 2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1 &= x(\mu); \\ 2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1 &= y(\mu); \\ 2\lambda \gamma_2 - \mu \beta_1 - (\nu_1 - \nu_2) \gamma_1 &= x(\lambda) - y(\nu_1); \\ 2\lambda \gamma_1 - \mu \beta_2 + (\nu_1 - \nu_2) \gamma_2 &= -x(\nu_2) + y(\lambda); \\ \gamma_1 \beta_1 - \gamma_2 \beta_2 - (\nu_1 - \nu_2) \mu &= -x(\beta_2) + y(\beta_1).\end{aligned}$$

Again the condition $\mu_u \mu_v \neq 0$ is equivalent to $(2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1)(2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1) \neq 0$. So, if $\mu_u \mu_v \neq 0$, then $\sqrt{E} = \frac{\mu_u}{2\mu \gamma_2 + \nu_1 \beta_2 - \lambda \beta_1}$; $\sqrt{G} = \frac{\mu_v}{2\mu \gamma_1 - \lambda \beta_2 + \nu_2 \beta_1}$.

In a similar way as in Section 5 we prove the following Bonnet-type theorem for spacelike surfaces in \mathbb{R}_1^4 with timelike mean curvature vector field.

Theorem 6.1. *Let $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ be smooth functions, defined in a domain $\mathcal{D}, \mathcal{D} \subset \mathbb{R}^2$, and satisfying the conditions*

$$\begin{aligned} \frac{\mu_u}{2\mu\gamma_2 + \nu_1\beta_2 - \lambda\beta_1} &> 0; \\ \frac{\mu_v}{2\mu\gamma_1 - \lambda\beta_2 + \nu_2\beta_1} &> 0; \\ -\gamma_1\sqrt{E}\sqrt{G} &= (\sqrt{E})_v; \\ -\gamma_2\sqrt{E}\sqrt{G} &= (\sqrt{G})_u; \\ -\nu_1\nu_2 + \lambda^2 - \mu^2 &= \frac{1}{\sqrt{E}}(\gamma_2)_u + \frac{1}{\sqrt{G}}(\gamma_1)_v - ((\gamma_1)^2 + (\gamma_2)^2); \\ 2\lambda\gamma_2 - \mu\beta_1 - (\nu_1 - \nu_2)\gamma_1 &= \frac{1}{\sqrt{E}}\lambda_u - \frac{1}{\sqrt{G}}(\nu_1)_v; \\ 2\lambda\gamma_1 - \mu\beta_2 + (\nu_1 - \nu_2)\gamma_2 &= -\frac{1}{\sqrt{E}}(\nu_2)_u + \frac{1}{\sqrt{G}}\lambda_v; \\ \gamma_1\beta_1 - \gamma_2\beta_2 - (\nu_1 - \nu_2)\mu &= -\frac{1}{\sqrt{E}}(\beta_2)_u + \frac{1}{\sqrt{G}}(\beta_1)_v, \end{aligned}$$

where $\sqrt{E} = \frac{\mu_u}{2\mu\gamma_2 + \nu_1\beta_2 - \lambda\beta_1}$, $\sqrt{G} = \frac{\mu_v}{2\mu\gamma_1 - \lambda\beta_2 + \nu_2\beta_1}$. Let $\{x_0, y_0, b_0, l_0\}$ be an orthonormal frame at a point $p_0 \in \mathbb{R}_1^4$. Then there exist a subdomain $\mathcal{D}_0 \subset \mathcal{D}$ and a unique spacelike surface $M^2 : z = z(u, v)$, $(u, v) \in \mathcal{D}_0$, passing through p_0 , with timelike mean curvature vector field, such that $\gamma_1, \gamma_2, \nu_1, \nu_2, \lambda, \mu, \beta_1, \beta_2$ are the geometric functions of M^2 and $\{x_0, y_0, b_0, l_0\}$ is the geometric frame of M^2 at the point p_0 .

7. EXAMPLES

In this section we shall apply our theory to a special class of spacelike surfaces in \mathbb{R}_1^4 . In [10] C. Moore studied general rotational surfaces in \mathbb{R}^4 . In [3, 5] we considered a special case of such surfaces, given by

$$(7.1) \quad \mathcal{M} : z(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cos \beta v, g(u) \sin \beta v),$$

where $u \in J \subset \mathbb{R}$, $v \in [0; 2\pi)$, $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2 + \beta^2 g^2 > 0$, $f'^2 + g'^2 > 0$, and α, β are positive constants. These surfaces are general rotational surfaces in the sense of C. Moore with plane meridian curves. Here we shall consider a class of spacelike surfaces in \mathbb{R}_1^4 which are analogous to (7.1).

Example 1. Let us consider the surface \mathcal{M}_1 parameterized by

$$\mathcal{M}_1 : z(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cosh \beta v, g(u) \sinh \beta v),$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying $\alpha^2 f^2(u) - \beta^2 g^2(u) > 0$, $f'^2(u) + g'^2(u) > 0$, $u \in J \subset \mathbb{R}$ and α, β are positive constants; $v \in [0; 2\pi)$. The tangent space of \mathcal{M}_1 is spanned by the vector fields

$$\begin{aligned} z_u &= (f'(u) \cos \alpha v, f'(u) \sin \alpha v, g'(u) \cosh \beta v, g'(u) \sinh \beta v), \\ z_v &= (-\alpha f(u) \sin \alpha v, \alpha f(u) \cos \alpha v, \beta g(u) \sinh \beta v, \beta g(u) \cosh \beta v). \end{aligned}$$

The coefficients of the first fundamental form of \mathcal{M}_1 are

$$E = f'^2(u) + g'^2(u); \quad F = 0; \quad G = \alpha^2 f^2(u) - \beta^2 g^2(u).$$

\mathcal{M}_1 is a spacelike surface in \mathbb{R}_1^4 . We choose the following normal frame field of \mathcal{M}_1 :

$$\begin{aligned} n_1 &= \frac{1}{\sqrt{f'^2 + g'^2}} (g' \cos \alpha v, g' \sin \alpha v, -f' \cosh \beta v, -f' \sinh \beta v); \\ n_2 &= \frac{1}{\sqrt{\alpha^2 f^2 - \beta^2 g^2}} (-\beta g \sin \alpha v, \beta g \cos \alpha v, \alpha f \sinh \beta v, \alpha f \cosh \beta v). \end{aligned}$$

We have $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$. Calculating the second partial derivatives of $z(u, v)$ we find the functions c_{ij}^k and get the coefficients L , M , N of the second fundamental form of \mathcal{M}_1 :

$$L = \frac{2\alpha\beta(gf' - fg')(g'f'' - f'g'')}{(f'^2 + g'^2)(\alpha^2 f^2 - \beta^2 g^2)}; \quad M = 0; \quad N = \frac{2\alpha\beta(gf' - fg')(\alpha^2 fg' + \beta^2 gf')}{(f'^2 + g'^2)(\alpha^2 f^2 - \beta^2 g^2)}.$$

Consequently, the invariants k , \varkappa , and K of \mathcal{M}_1 are expressed as follows:

$$\begin{aligned} k &= \frac{4\alpha^2\beta^2(gf' - fg')^2(g'f'' - f'g'')(\alpha^2 fg' + \beta^2 gf')}{(f'^2 + g'^2)^3(\alpha^2 f^2 - \beta^2 g^2)^3}; \\ \varkappa &= \frac{\alpha\beta(gf' - fg')}{(f'^2 + g'^2)^2(\alpha^2 f^2 - \beta^2 g^2)^2} [(\alpha^2 f^2 - \beta^2 g^2)(g'f'' - f'g'') + (f'^2 + g'^2)(\alpha^2 fg' + \beta^2 gf')]; \\ K &= \frac{-(\alpha^2 f^2 - \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f'' - f'g'') + \alpha^2\beta^2(f'^2 + g'^2)(gf' - fg')^2}{(f'^2 + g'^2)^2(\alpha^2 f^2 - \beta^2 g^2)^2}. \end{aligned}$$

The mean curvature vector field H is collinear with n_1 , and hence \mathcal{M}_1 is a spacelike surface whose mean curvature vector at any point is a non-zero spacelike vector. Note that \mathcal{M}_1 is parameterized by principal parameters (u, v) . Denoting $x = \frac{z_u}{\sqrt{f'^2(u) + g'^2(u)}}$, $y = \frac{z_v}{\sqrt{\alpha^2 f^2(u) - \beta^2 g^2(u)}}$ we obtain the geometric invariant functions in the Frenet-type derivative formulas of \mathcal{M}_1 :

$$\begin{aligned} \gamma_1 &= 0; & \gamma_2 &= -\frac{\alpha^2 ff' - \beta^2 gg'}{\sqrt{f'^2 + g'^2}(\alpha^2 f^2 - \beta^2 g^2)}; \\ \nu_1 &= \frac{g'f'' - f'g''}{(f'^2 + g'^2)^{\frac{3}{2}}}; & \nu_2 &= -\frac{\alpha^2 fg' + \beta^2 gf'}{\sqrt{f'^2 + g'^2}(\alpha^2 f^2 - \beta^2 g^2)}; \\ \lambda &= 0; & \mu &= \frac{\alpha\beta(gf' - fg')}{\sqrt{f'^2 + g'^2}(\alpha^2 f^2 - \beta^2 g^2)}; \\ \beta_1 &= 0; & \beta_2 &= \frac{\alpha\beta(ff' + gg')}{\sqrt{f'^2 + g'^2}(\alpha^2 f^2 - \beta^2 g^2)}. \end{aligned}$$

Since the invariant λ is zero, the general rotational surface \mathcal{M}_1 is a Chen surface. In [8] C. Houh considered a more general class of surfaces of rotational type in \mathbb{R}_1^4 and found a subclass of Chen surfaces.

In the special case when $f(u) = \cos u$, $g(u) = \sin u$, $\alpha = \beta = 1$ we obtain a spacelike surface lying on De Sitter space $S_1^3 = \{x \in \mathbb{R}_1^4; \langle x, x \rangle = 1\}$ with invariants

$$k = -\frac{4}{\cos^2 2u}; \quad \varkappa = 0; \quad K = \frac{\cos^2 2u + 1}{\cos^2 2u}.$$

This is an example of a spacelike surface with flat normal connection and spacelike mean curvature vector field.

Example 2. Now we shall consider the surface \mathcal{M}_2 parameterized by

$$\mathcal{M}_2 : z(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \sinh \beta v, g(u) \cosh \beta v),$$

where $f(u)$ and $g(u)$ are smooth functions, satisfying $f'^2(u) - g'^2(u) > 0$, $\alpha^2 f^2(u) + \beta^2 g^2(u) > 0$, $u \in J \subset \mathbb{R}$ and α, β are positive constants; $v \in [0; 2\pi)$. The coefficients of the first fundamental form of \mathcal{M}_2 are

$$E = f'^2(u) - g'^2(u); \quad F = 0; \quad G = \alpha^2 f^2(u) + \beta^2 g^2(u),$$

hence \mathcal{M}_2 is a spacelike surface in \mathbb{R}_1^4 . We choose the following normal frame field of \mathcal{M}_2 :

$$n_1 = \frac{1}{\sqrt{\alpha^2 f^2 + \beta^2 g^2}} (\beta g \sin \alpha v, -\beta g \cos \alpha v, \alpha f \cosh \beta v, \alpha f \sinh \beta v);$$

$$n_2 = \frac{1}{\sqrt{f'^2 - g'^2}} (g' \cos \alpha v, g' \sin \alpha v, f' \sinh \beta v, f' \cosh \beta v).$$

We have $\langle n_1, n_1 \rangle = 1$, $\langle n_2, n_2 \rangle = -1$. Calculating the coefficients L , M , N of the second fundamental form we obtain that the invariants k , \varkappa , and K of \mathcal{M}_2 are expressed by the functions $f(u)$, $g(u)$ and their derivatives as follows:

$$k = \frac{4\alpha^2\beta^2(gf' - fg')(g'f'' - f'g'')(\alpha^2 fg' + \beta^2 gf')}{(f'^2 - g'^2)^3(\alpha^2 f^2 + \beta^2 g^2)^3};$$

$$\varkappa = \frac{\alpha\beta(gf' - fg')}{(f'^2 - g'^2)^2(\alpha^2 f^2 + \beta^2 g^2)^2} [(\alpha^2 f^2 + \beta^2 g^2)(g'f'' - f'g'') + (f'^2 - g'^2)(\alpha^2 fg' + \beta^2 gf')];$$

$$K = \frac{(\alpha^2 f^2 + \beta^2 g^2)(\alpha^2 fg' + \beta^2 gf')(g'f'' - f'g'') - \alpha^2\beta^2(f'^2 - g'^2)(gf' - fg')^2}{(f'^2 - g'^2)^2(\alpha^2 f^2 + \beta^2 g^2)^2}.$$

In this example the mean curvature vector field H is collinear with n_2 , and hence \mathcal{M}_2 is a spacelike surface with timelike mean curvature vector field. We note that \mathcal{M}_2 is parameterized by principal parameters (u, v) . The geometric invariant functions in the Frenet-type derivative formulas of \mathcal{M}_2 are given below:

$$\begin{aligned} \gamma_1 &= 0; & \gamma_2 &= -\frac{\alpha^2 f f' + \beta^2 g g'}{\sqrt{f'^2 - g'^2}(\alpha^2 f^2 + \beta^2 g^2)}; \\ \nu_1 &= \frac{g' f'' - f' g''}{(f'^2 - g'^2)^{\frac{3}{2}}}; & \nu_2 &= -\frac{\alpha^2 f g' + \beta^2 g f'}{\sqrt{f'^2 - g'^2}(\alpha^2 f^2 + \beta^2 g^2)}; \\ \lambda &= 0; & \mu &= \frac{\alpha\beta(f g' - g f')}{\sqrt{f'^2 - g'^2}(\alpha^2 f^2 + \beta^2 g^2)}; \\ \beta_1 &= 0; & \beta_2 &= \frac{\alpha\beta(g g' - f f')}{\sqrt{f'^2 - g'^2}(\alpha^2 f^2 + \beta^2 g^2)}. \end{aligned}$$

The surface \mathcal{M}_2 is a spacelike Chen surface, since the invariant λ is zero.

If we choose $f(u) = \sinh u$, $g(u) = \cosh u$, $\alpha = \beta = 1$ we obtain a spacelike surface lying on the unit hyperbolic sphere $H_1^3 = \{x \in \mathbb{R}_1^4; \langle x, x \rangle = -1\}$ with invariants

$$k = -\frac{4}{\cosh^2 2u}; \quad \varkappa = 0; \quad K = -\frac{\cosh^2 2u + 1}{\cosh^2 2u}.$$

This is a spacelike surface with flat normal connection and timelike mean curvature vector field.

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